The inverse of a 2×2 matrix

Introduction

Once you know how to multiply matrices it is natural to ask whether they can be divided. The answer is no. However, by defining another matrix called the **inverse matrix** it is possible to work with an operation which plays a similar role to division. In this leaflet we explain what is meant by an inverse matrix and how the inverse of a 2×2 matrix is calculated.

1. The inverse of a 2×2 matrix

The **inverse** of a 2×2 matrix A, is another 2×2 matrix denoted by A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I$$

where I is the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. That is, multiplying a matrix by its inverse produces an identity matrix. Note that in this context A^{-1} does not mean $\frac{1}{A}$.

Not all 2×2 matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be **singular**. Only non-singular matrices have inverses.

2. A simple formula for the inverse

In the case of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a simple formula exists to find its inverse:

$$\text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{then} \quad A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

Find the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(2) - (1)(4)}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

This could be written as

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

You should check that this answer is correct by performing the matrix multiplication AA^{-1}. The result should be the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example

Find the inverse of the matrix $A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$.

Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(1) - (4)(-3)} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{14} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

This can be written

$$A^{-1} = \begin{pmatrix} 1/14 & -4/14 \\ 3/14 & 2/14 \end{pmatrix} = \begin{pmatrix} 1/14 & -2/7 \\ 3/14 & 1/7 \end{pmatrix}$$

although it is quite permissible to leave the factor $\frac{1}{14}$ at the front of the matrix.

Exercises

1. Find the inverse of $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$.

2. Explain why the inverse of the matrix $\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$ cannot be calculated.

3. Show that $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ is the inverse of $\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$.

Answers

1. $A^{-1} = \frac{1}{-13} \begin{pmatrix} 2 & -5 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{1}{13} \end{pmatrix}$.

2. The determinant of the matrix is zero, that is, it is singular and so has no inverse.

